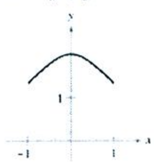
Unit 4 day 4

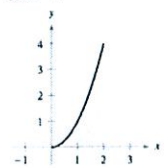
Warm up

Set up a definite integral that yields the area of the region. (do no evaluate the integral)

$f(x) = \frac{4}{x^2 + 2}$



$f(x) = x^2$



$\int_{-1}^{1} \frac{4}{x^2+2}\, dx$ $\int_{0}^{2} x^2\, dx$

Homework Check

- Evaluate the definite integral by the limit definition. $\int_{-2}^{3} x\, dx$
- Sketch the region given by the definite integral. Then use a geometric formula to evaluate the integral. $\int_{0}^{6} (6 - x)\, dx$

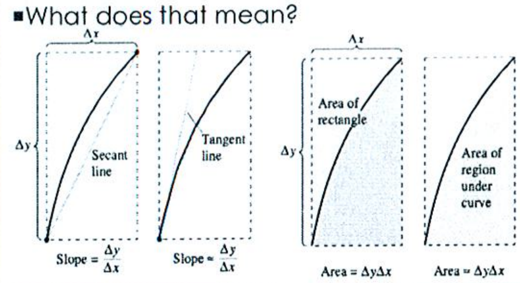
Fundamental Theorem of Calculus

The Fundamental Theorem of Calculus

You have now been introduced to the two major branches of calculus: differential calculus (introduced with the tangent line problem) and integral calculus (introduced with the area problem). At this point, these two problems might seem unrelated—but there is a very close connection. The connection was discovered independently by Isaac Newton and Gottfried Leibniz and is stated in a theorem that is appropriately called the **Fundamental Theorem of Calculus.**

Differentiation and Integration are inverse operations

- What does that mean?



inverses

So in short Definite integrals can make the area under a curve problems much easier

THEOREM 4.9 THE FUNDAMENTAL THEOREM OF CALCULUS

If a function f is continuous on the closed interval $[a, b]$ and F is an antiderivative of f on the interval $[a, b]$, then

$$\int_a^b f(x)\, dx = F(b) - F(a).$$

Two important notes

$$\int_a^b f(x)\, dx = F(x)\Big]_a^b$$
$$= F(b) - F(a)$$

For instance, to evaluate $\int_1^3 x^3\, dx$, you can write

$$\int_1^3 x^3\, dx = \frac{x^4}{4}\Big]_1^3 = \frac{3^4}{4} - \frac{1^4}{4} = \frac{81}{4} - \frac{1}{4} = 20.$$

It is not necessary to include a constant of integration C in the antiderivative because

$$\int_a^b f(x)\, dx = \Big[F(x) + C\Big]_a^b$$

10/16/14

Evaluate each definite integral [redacted]

$$\int_1^2 (x^2-3)\,dx = \frac{x^3}{3} - 3x \Big|_1^2 = \left[\frac{2^3}{3} - 3(2)\right] - \left[\frac{1^3}{3} - 3(1)\right] =$$

$$\left[\frac{8}{3} - 6\right] - \left[\frac{1}{3} - 3\right] = \boxed{\frac{-2}{3}}$$

$$\int_1^4 3\sqrt{x}\,dx$$

$$3\int_1^4 x^{1/2}\,dx = \frac{3x^{3/2}}{3/2}\Big|_1^4$$

$$2x^{3/2}\Big|_1^4 = 2(4)^{3/2} - 2(1)^{3/2} = \boxed{14}$$

Evaluate each definite integral [redacted]

$$\int_0^2 (2x^2-3x+2)\,dx = \frac{2x^3}{3} - \frac{3x^2}{2} + 2x \Big|_0^2$$

$$\left[\frac{2(2)^3}{3} - \frac{3(2)^2}{2} + 2(2)\right] - \left[\frac{2(0)^3}{3} - \frac{3(0)^2}{2} + 2(0)\right] = \frac{16}{3} - 6 + 4 - 0$$

$$\boxed{\frac{10}{3}}$$

$$\int_0^{\pi/4} \sec^2 x\,dx$$

$$\tan x \Big|_0^{\pi/4} \quad \tan\frac{\pi}{4} - \tan 0$$

$$1 - 0 = \boxed{1}$$

Evaluate each definite integral [redacted]

$$\int_2^5 (-3x+4)\,dx \quad \frac{-3x^2}{2} + 4x \Big|_2^5$$

$$\left[\frac{-3(5)^2}{2} + 4(5)\right] - \left[\frac{-3(2)^2}{2} + 4(2)\right] = \left(\frac{-75}{2} + 20\right) - (6+8)$$

$$\boxed{-19.5}$$

$$\int_0^{\pi/4} \cos(x)\,dx$$

$$\sin x \Big|_0^{\pi/4}$$

$$\sin\frac{\pi}{4} - \sin 0$$

$$\frac{\sqrt{2}}{2} - 0 = \boxed{\frac{\sqrt{2}}{2}}$$

Evaluate each definite integral [redacted]

* absolute value

* split up abs. value

$$\int_0^2 |2x-1|\,dx$$

$$|2x-1| = \begin{cases} -2x+1 & x < 1/2 \\ 2x-1 & x > 1/2 \end{cases}$$

$$\int_0^{1/2} -2x+1\,dx + \int_{1/2}^2 2x-1\,dx$$

$$-x^2+x\Big|_0^{1/2} + x^2 - x\Big|_{1/2}^2 = \boxed{\frac{5}{2}}$$

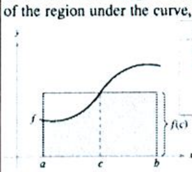
The Mean Value Theorem for Integrals

In Section 4.2, you saw that the area of a region under a curve is greater than the area of an inscribed rectangle and less than the area of a circumscribed rectangle. The Mean Value Theorem for Integrals states that somewhere "between" the inscribed and circumscribed rectangles there is a rectangle whose area is precisely equal to the area of the region under the curve, as shown in Figure 4.29.

THEOREM 4.10 MEAN VALUE THEOREM FOR INTEGRALS

If f is continuous on the closed interval $[a, b]$, then there exists a number c in the closed interval $[a, b]$ such that

$$\int_a^b f(x)\,dx = f(c)(b-a).$$



Mean value rectangle:

$$f(c)(b-a) = \int_a^b f(x)\,dx$$

$$\frac{\int_a^b f(x)\,dx}{(b-a)} = \frac{f(c)\cancel{(b-a)}}{\cancel{(b-a)}}$$

$$\frac{\int_a^b f(x)\,dx}{(b-a)} = f(c)$$

Find the value(s) of c guaranteed by the Mean Value Theorem for integrals for the function given the interval [redacted]

$$f(x) = 4x^2 + 3 \qquad [0, 3]$$

$$\int_0^3 (4x^2+3)\,dx = \frac{4x^3}{3} + 3x \Big|_0^3$$

$$\left[\frac{4(3)^3}{3} + 3(3)\right] - (0)$$

$$36 + 9 - 0 = 45$$

$$\frac{45}{3-0} = \quad 15 = f(c)$$

$$15 - 3 = 4c^2 + 3 - 3$$

$$\frac{12}{4} = \frac{4c^2}{4}$$

$$\sqrt{3} = \sqrt{c^2}$$

$$\boxed{c = \sqrt{3}}$$

$\int_a^b f(x)dx = f(c)(b-a)$

$\frac{-9}{2x^2}\Big|_1^3 = \frac{-9}{2(3)^2} - \frac{-9}{2(1)^2} = -\frac{1}{2} + \frac{9}{2} = 4$

10/16/14

$\frac{\int_1^3 9x^{-3}dx}{2}$ $\quad \frac{9x^{-2}}{-2}\Big|_1^3 = $

Find the value(s) of c guaranteed by the Mean Value Theorem for integrals for the function given the interval

$$f(x) = \frac{9}{x^3} \qquad [1, 3]$$

$\frac{4}{2} = f(c)$

$\frac{2}{1} = \frac{9}{x^3}$ $\quad \frac{9}{2} = \frac{2x^3}{2}$

$\sqrt[3]{\frac{9}{2}} = \sqrt[3]{x^3}$ $\quad \boxed{x = \sqrt[3]{\frac{9}{2}}}$

Average Value of a Function

The value of $f(c)$ given in the Mean Value Theorem for Integrals is called the **average value** of f on the interval $[a, b]$.

DEFINITION OF THE AVERAGE VALUE OF A FUNCTION ON AN INTERVAL

If f is integrable on the closed interval $[a, b]$, then the **average value** of f on the interval is

$$\frac{1}{b-a}\int_a^b f(x)\,dx.$$

Find the average value of the function over the given interval

- $f(x) = 3x^2 - 2x \quad [1, 4]$

$\frac{1}{4-1}\int_1^4 (3x^2 - 2x)dx$

$\frac{1}{3}\left[\frac{3x^3}{3} - \frac{2x^2}{2}\right]_1^4$

$\frac{1}{3}\left[x^3 - x^2\right]_1^4$ $\quad \frac{1}{3}\left[\left[(4)^3 - (4)^2\right] - \left(1^3 - 1^2\right)\right]$

$\frac{1}{3}\left(64 - 16 - (1-1)\right)$ $\quad \frac{1}{3}(48) = \boxed{16}$

Find the average value of the function over the given interval

- $f(x) = \cos(x) \quad [0, \pi/2]$

$\frac{1}{\frac{\pi}{2} - 0}\int_0^{\frac{\pi}{2}} \cos x \, dx$

$\frac{1}{\frac{\pi}{2}}\left(\sin(x)\right)_0^{\frac{\pi}{2}} = \frac{2}{\pi}\left[\sin\frac{\pi}{2} - \sin 0\right]$

$\frac{2}{\pi}\left[1 - 0\right] = \boxed{\frac{2}{\pi}}$

Find the average value of the function over the given interval

$$f(x) = \frac{4(x^2+1)}{x^2} \qquad [1, 3]$$

$\frac{1}{3-1}\int_1^3 \left(\frac{4x^2}{x^2} + \frac{4}{x^2}\right)dx$

$\frac{1}{2}\int_1^3 (4 + 4x^{-2})dx$

$\frac{1}{2}\left[4x + \frac{4x^{-1}}{-1}\right]_1^3 = \frac{1}{2}\left[4x + \frac{-4}{x}\right]_1^3$

$\frac{1}{2}\left[\left(4(3) + \frac{-4}{3}\right) - \left(4(1) + \frac{-4}{1}\right)\right] = \frac{1}{2}\left(\frac{22}{3} - 0\right) = \frac{22}{6} = \boxed{\frac{11}{3} \approx 3.\bar{6}}$

Homework

- Page 284
- 7 – 15 (odd)
- Page 285
- 45 – 51 (odd)

EXERCISES FOR SECTION 4.4

Graphical Reasoning **In Exercises 1–4, use a graphing utility to graph the integrand. Use the graph to determine whether the definite integral is positive, negative, or zero.**

1. $\displaystyle\int_0^{\pi} \frac{4}{x^2+1}\,dx$
2. $\displaystyle\int_0^{\pi} \cos x\,dx$
3. $\displaystyle\int_{-2}^{2} x\sqrt{x^2+1}\,dx$
4. $\displaystyle\int_{-2}^{2} x\sqrt{2-x}\,dx$

In Exercises 5–26, evaluate the definite integral of the algebraic function. Use a graphing utility to verify your result.

5. $\displaystyle\int_0^1 2x\,dx$
6. $\displaystyle\int_2^7 3\,dv$
7. $\displaystyle\int_{-1}^0 (x-2)\,dx$
8. $\displaystyle\int_2^5 (-3v+4)\,dv$
9. $\displaystyle\int_{-1}^1 (t^2-2)\,dt$
10. $\displaystyle\int_1^3 (3x^2+5x-4)\,dx$
11. $\displaystyle\int_0^1 (2t-1)^2\,dt$
12. $\displaystyle\int_{-1}^1 (t^3-9t)\,dt$
13. $\displaystyle\int_1^2 \left(\frac{3}{x^2}-1\right)dx$
14. $\displaystyle\int_{-2}^{-1} \left(u-\frac{1}{u^2}\right)du$
15. $\displaystyle\int_1^4 \frac{u-2}{\sqrt{u}}\,du$
16. $\displaystyle\int_{-3}^3 v^{1/3}\,dv$
17. $\displaystyle\int_{-1}^1 (\sqrt[3]{t}-2)\,dt$
18. $\displaystyle\int_1^8 \sqrt{\frac{2}{x}}\,dx$
19. $\displaystyle\int_0^1 \frac{x-\sqrt{x}}{3}\,dx$
20. $\displaystyle\int_0^2 (2-t)\sqrt{t}\,dt$
21. $\displaystyle\int_{-1}^0 (t^{1/3}-t^{2/3})\,dt$
22. $\displaystyle\int_{-8}^{-1} \frac{x-x^2}{2\sqrt[3]{x}}\,dx$
23. $\displaystyle\int_0^3 |2x-3|\,dx$
24. $\displaystyle\int_1^4 (3-|x-3|)\,dx$
25. $\displaystyle\int_0^3 |x^2-4|\,dx$
26. $\displaystyle\int_0^4 |x^2-4x+3|\,dx$

In Exercises 27–32, evaluate the definite integral of the trigonometric function. Use a graphing utility to verify your result.

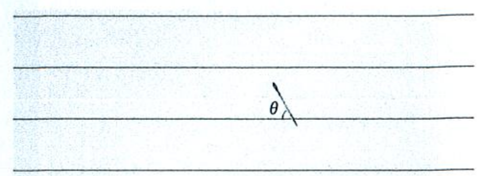
27. $\displaystyle\int_0^{\pi} (1+\sin x)\,dx$
28. $\displaystyle\int_0^{\pi/4} \frac{1-\sin^2\theta}{\cos^2\theta}\,d\theta$
29. $\displaystyle\int_{-\pi/6}^{\pi/6} \sec^2 x\,dx$
30. $\displaystyle\int_{\pi/4}^{\pi/2} (2-\csc^2 x)\,dx$
31. $\displaystyle\int_{-\pi/3}^{\pi/3} 4\sec\theta\tan\theta\,d\theta$
32. $\displaystyle\int_{-\pi/2}^{\pi/2} (2t+\cos t)\,dt$

33. ***Depreciation*** A company purchases a new machine for which the rate of depreciation is $dV/dt = 10{,}000(t-6)$, $0 \le t \le 5$, where V is the value of the machine after t years. Set up and evaluate the definite integral that yields the total loss of value of the machine over the first 3 years.

34. ***Buffon's Needle Experiment*** A horizontal plane is ruled with parallel lines 2 inches apart. If a 2-inch needle is tossed randomly onto the plane, the probability that the needle will touch a line is

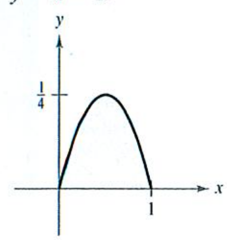
$$P = \frac{2}{\pi}\int_0^{\pi/2} \sin\theta\,d\theta$$

where θ is the acute angle between the needle and any one of the parallel lines. Find this probability.

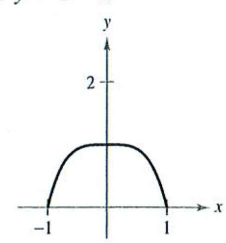


In Exercises 35–40, determine the area of the indicated region.

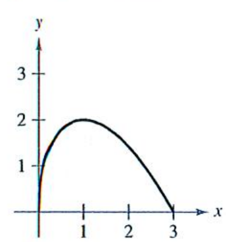
35. $y = x - x^2$



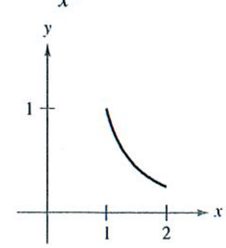
36. $y = 1 - x^4$



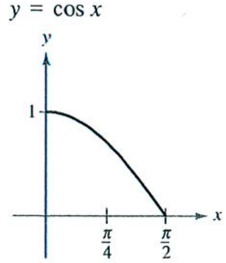
37. $y = (3-x)\sqrt{x}$



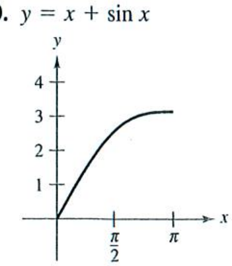
38. $y = \dfrac{1}{x^2}$



39. $y = \cos x$



40. $y = x + \sin x$



In Exercises 41–44, find the area of the region bounded by the graphs of the equations.

41. $y = 3x^2 + 1$, $x = 0$, $x = 2$, $y = 0$
42. $y = 1 + \sqrt[3]{x}$, $x = 0$, $x = 8$, $y = 0$
43. $y = x^3 + x$, $x = 2$, $y = 0$
44. $y = -x^2 + 3x$, $y = 0$

In Exercises 45–48, find the value(s) of c guaranteed by the Mean Value Theorem for Integrals for the function over the indicated interval.

Function	*Interval*
45. $f(x) = x - 2\sqrt{x}$	$[0, 2]$
46. $f(x) = \dfrac{9}{x^3}$	$[1, 3]$
47. $f(x) = 2\sec^2 x$	$[-\pi/4, \pi/4]$
48. $f(x) = \cos x$	$[-\pi/3, \pi/3]$

In Exercises 49–52, find the average value of the function over the interval and all values of x in the interval for which the function equals its average value.

Function	*Interval*
49. $f(x) = 4 - x^2$	$[-2, 2]$
50. $f(x) = \dfrac{4(x^2+1)}{x^2}$	$[1, 3]$
51. $f(x) = \sin x$	$[0, \pi]$
52. $f(x) = \cos x$	$[0, \pi/2]$

Getting at the Concept

53. State the Fundamental Theorem of Calculus.

54. The graph of f is given in the figure.

(a) Evaluate $\displaystyle\int_1^7 f(x)\,dx$.

(b) Determine the average value of f on the interval $[1, 7]$.

(c) Determine the answers to parts (a) and (b) if the graph is translated two units upward.

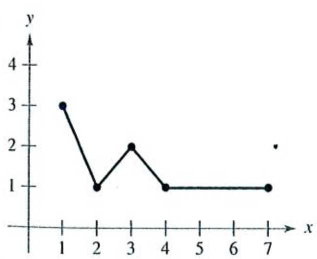


Figure for 54

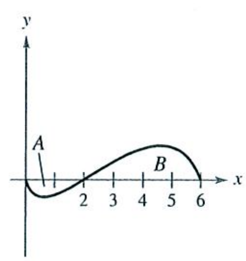


Figure for 55–60

In Exercises 55–60, use the graph of f shown in the figure. The shaded region A has an area of 1.5, and $\int_0^6 f(x)\,dx = 3.5$. Use this information to fill in the blanks.

55. $\displaystyle\int_0^2 f(x)\,dx =$

56. $\displaystyle\int_2^6 f(x)\,dx =$

57. $\displaystyle\int_0^6 |f(x)|\,dx =$

58. $\displaystyle\int_0^2 -2f(x)\,dx =$

59. $\displaystyle\int_0^6 [2 + f(x)]\,dx =$

60. The average value of f over the interval $[0, 6]$ is .

61. ***Force*** The force F (in newtons) of a hydraulic cylinder in a press is proportional to the square of $\sec x$, where x is the distance (in meters) that the cylinder is extended in its cycle. The domain of F is $[0, \pi/3]$, and $F(0) = 500$.

(a) Find F as a function of x.

(b) Find the average force exerted by the press over the interval $[0, \pi/3]$.

62. ***Blood Flow*** The velocity v of the flow of blood at a distance r from the central axis of an artery of radius R is

$$v = k(R^2 - r^2)$$

where k is the constant of proportionality. Find the average rate of flow of blood along a radius of the artery. (Use 0 and R as the limits of integration.)

63. ***Respiratory Cycle*** The volume V in liters of air in the lungs during a 5-second respiratory cycle is approximated by the model

$$V = 0.1729t + 0.1522t^2 - 0.0374t^3$$

where t is the time in seconds. Approximate the average volume of air in the lungs during one cycle.

64. ***Average Profit*** A company introduces a new product, and the profit in thousands of dollars over the first 6 months is approximated by the model

$$P = 5(\sqrt{t} + 30), \qquad t = 1, 2, 3, 4, 5, 6.$$

(a) Use the model to complete the table and use the entries to calculate (arithmetically) the average profit over the first 6 months.

t	1	2	3	4	5	6
P						

(b) Find the average value of the profit function by integration and compare the result with that in part (a). (Integrate over the interval $[0.5, 6.5]$.)

(c) What, if any, is the advantage of using the approximation of the average given by the definite integral? (Note that the integral approximation utilizes all real values of t in the interval rather than just integers.)

65. ***Average Sales*** A company fit a model to the monthly sales data of a seasonal product. The model is

$$S(t) = \frac{t}{4} + 1.8 + 0.5\sin\left(\frac{\pi t}{6}\right), \qquad 0 \le t \le 24$$

where S is sales (in thousands) and t is time in months.

(a) Use a graphing utility to graph $f(t) = 0.5\sin(\pi t/6)$ for $0 \le t \le 24$. Use the graph to explain why the average value of $f(t)$ is 0 over the interval.

(b) Use a graphing utility to graph $S(t)$ and the line $g(t) = t/4 + 1.8$ in the same viewing window. Use the graph and the result of part (a) to explain why g is called the *trend line*.

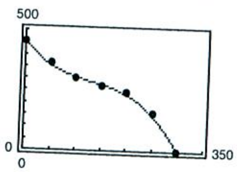
81. Suppose there are n rows in the figure. The stars on the left total $1 + 2 + \cdots + n$, as do the stars on the right. There are $n(n + 1)$ stars in total. So,

$$2[1 + 2 + \cdots + n] = n(n + 1)$$

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2}.$$

83. (a) $y = (-4.09 \times 10^{-5})x^3 + 0.016x^2 - 2.67x + 452.9$

(b)



(c) 76,897 square feet

Section 4.3 (page 272)

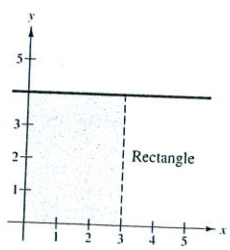
1. $2\sqrt{3} \approx 3.464$ **3.** 36 **5.** 0 **7.** $\frac{10}{3}$

9. $\int_{-1}^{5} (3x + 10)\, dx$ **11.** $\int_{0}^{3} \sqrt{x^2 + 4}\, dx$

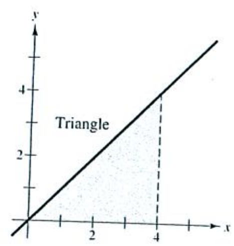
13. $\int_{0}^{5} 3\, dx$ **15.** $\int_{-4}^{4} (4 - |x|)\, dx$ **17.** $\int_{-2}^{2} (4 - x^2)\, dx$

19. $\int_{0}^{\pi} \sin x\, dx$ **21.** $\int_{0}^{2} y^3\, dy$

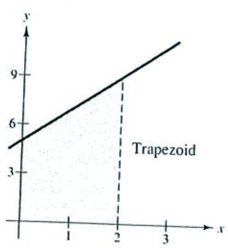
23. $A = 12$



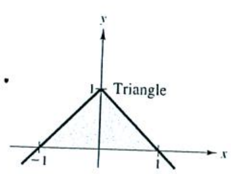
25. $A = 8$



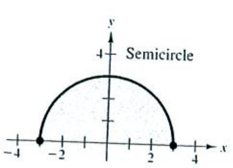
27. $A = 14$



29. $A = 1$



31. $A = \frac{9\pi}{2}$



33. -6 **35.** 24 **37.** -10 **39.** 16

41. (a) 13 (b) -10 (c) 0 (d) 30

43. (a) 8 (b) -12 (c) -4 (d) 30

45. (a) $-\pi$ (b) 4 (c) $-(1 + 2\pi)$ (d) $3 - 2\pi$ (e) $5 + 2\pi$ (f) $23 - 2\pi$

47. $\sum_{i=1}^{n} f(x_i)\,\Delta x > \int_{1}^{5} f(x)\, dx$ **49.** $\sum_{i=1}^{n} f(x_i)\,\Delta x < \int_{1}^{5} f(x)\, dx$

51. No. There is a discontinuity at $x = 4$.

53. a **55.** d

57.

n	4	8	12	16	20
$L(n)$	3.6830	3.9956	4.0707	4.1016	4.1177
$M(n)$	4.3082	4.2076	4.1838	4.1740	4.1690
$R(n)$	3.6830	3.9956	4.0707	4.1016	4.1177

59.

n	4	8	12	16	20
$L(n)$	0.5890	0.6872	0.7199	0.7363	0.7461
$M(n)$	0.7854	0.7854	0.7854	0.7854	0.7854
$R(n)$	0.9817	0.8836	0.8508	0.8345	0.8247

61. True **63.** True

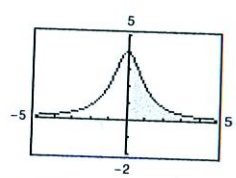
65. False: $\int_{0}^{2} (-x)\, dx = -2$ **67.** 272

69. No. No matter how small the subintervals, the number of both rational and irrational numbers within each subinterval is infinite and $f(c_i) = 0$ or $f(c_i) = 1$.

71. $\frac{1}{3}$

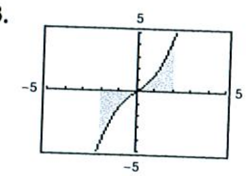
Section 4.4 (page 284)

1.



Positive

3.



Zero

5. 1 **7.** $-\frac{5}{2}$ **9.** $-\frac{10}{3}$ **11.** $\frac{1}{3}$ **13.** $\frac{1}{2}$ **15.** $\frac{2}{3}$

17. -4 **19.** $-\frac{1}{18}$ **21.** $-\frac{27}{20}$ **23.** $\frac{9}{2}$ **25.** $\frac{23}{3}$

27. $\pi + 2$ **29.** $\frac{2\sqrt{3}}{3}$ **31.** 0

33. $\int_{0}^{3} 10{,}000(t - 6)\, dt = -\$135{,}000$ **35.** $\frac{1}{6}$

37. $\frac{12\sqrt{3}}{5}$ **39.** 1 **41.** 10 **43.** 6

45. 0.4380, 1.7908 **47.** $\pm\arccos \frac{\sqrt{\pi}}{2} \approx \pm 0.4817$

49. Average value $= \frac{8}{3}$

$x = \pm\frac{2\sqrt{3}}{3} \approx \pm 1.155$

51. Average value $= \frac{2}{\pi}$

$x \approx 0.690, x \approx 2.451$

53. The Fundamental Theorem of Calculus states that if a function f is continuous on $[a, b]$ and F is an antiderivative of f on $[a, b]$, then $\int_a^b f(x)\, dx = F(b) - F(a)$.