Unit 6 day 4

Warm Up
Find the volume of the solid generated by revolving the region bounded by the graphs of the equations about the y-axis. (A sketch would be very useful)

\[ \pi \int_{-2}^{2} \left[ (6^2 - (\frac{6}{y})^2) \right] \, dy \]

\[ \pi \int_{-2}^{2} \left( 36 - \frac{36}{y^2} \right) \, dy = \pi \left[ 36 \ln y + \frac{36}{y} \right]_{-2}^{2} \]

\[ 2 \pi \left( \ln 2 - \ln 2 \right) = 2 \pi \ln 1 = 0 \]

Homework Check
Find the area of the region bounded by the graphs of the algebraic functions below

- \( x = y^2 + 1 \)
- \( x = 5 \)
- \( y = -1 \)
- \( y = 2 \)

Solids with known cross sections
With the disk method you can find the volume of a solid having a circular cross section.

This method can be generalized to solids of any shape, as long as you know a formula for the area of an arbitrary cross section.

Some common cross sections are squares, rectangles, triangles, semicircles, and trapezoids.

Common Areas
- Square \( A = x^2 \)
- Circle \( A = \pi r^2 \)
- Triangles \( A = \frac{1}{2} bh \)

I. General Slicing Method
The volume of a geometric solid with uniform cross sectional area is defined as the area of the base times the height.

\[ V = A \times h \]

Volumes of Solids with Known Cross Sections
1. For cross sections of area \( A(x) \) taken perpendicular to the x-axis,
   Volume = \( \int A(x) \, dx \)

2. For cross sections of area \( A(y) \) taken perpendicular to the y-axis,
   Volume = \( \int A(y) \, dy \)
1. Sketch the solid and a typical cross section.
2. Find a formula for \( A(x) \).
   (When you multiply \( A(x) \) by \( dx \), you will have the third length necessary to find the volume)
3. Find the limits of integration.
4. Integrate \( A(x) \) to find volume.

\[
A(x) = (1 - x^2)^2
\]

**Volume of a parabolic cube**
- Let \( R \) be the region in the first quadrant bounded by the coordinate axes and the curve \( y = 1 - x^2 \).
- Each cross section perpendicular to the \( x \) axis is a square.

\[
\int_0^1 (1 - x^2)^2 \, dx
\]
\[
\int_0^1 (1 - 2x^2 - x^4) \, dx
\]
\[
\left[ x - \frac{2x^3}{3} - \frac{x^5}{5} \right]_0^1 = \frac{8}{15}
\]

**Area of equilateral triangle**
- \( A = \frac{1}{2} \cdot b \cdot h \)
- \( a^2 + b^2 = c^2 \)
- \( a^2 + \left( \frac{\sqrt{3}}{2} \cdot x - 2 \right)^2 = (x^2 - 4)^2 \)
- \( a^2 + \left( \frac{\sqrt{3}}{4} \cdot x - 2 \right)^2 = (x^2 - 4)^2 \)
- \( a^2 + \frac{\sqrt{3}}{4} \cdot x - 2 = x^2 - 4 \)
- \( a^2 = \frac{3}{4} \cdot x^4 - 6x^2 + 16 \)
- \( 4a^2 = 3 \left( x^2 - 4 \right)^2 \)
- \( a^2 = \frac{3}{4} \left( x^2 - 4 \right) \)
- \( a = \sqrt{\frac{3}{4} \left( x^2 - 4 \right)} \)

The curve \( y = x^2 \) from 0 to 2 is pictured below.
A solid is constructed such that the cross sections perpendicular to the \( x \)-axis are squares.

\[
A(x) = \left( x^2 \right)^2
\]
\[
\int_0^1 x^2 \, dx = \frac{x^5}{5} \bigg|_0^1 = \frac{32}{5}
\]

Let \( R \) be the region bounded by the graph of \( y = 4 - x^2 \) and \( y = 0 \)
Find the volume of the solid with base of region \( R \) and cross sections perpendicular to the \( x \)-axis and square in shape.

\[
\int_{-2}^2 (4 - x^2) \, dx
\]
\[
\int_{-2}^2 (x^4 - 8x^2 + 16) \, dx
\]
\[
\left[ \frac{x^5}{5} - \frac{8x^3}{3} + 16x \right]_{-2}^2 = 34.13
\]

Let \( R \) be the region bounded by the graph of \( y = x^2 - 4 \) and the \( x \)-axis
between \( x = 0 \) and \( x = 2 \)
Find the volume of the solid with base of region \( R \) and cross sections perpendicular to the \( x \)-axis if the cross sections are equilateral triangles.

\[
\int_0^2 \sqrt{3} \cdot \left( x^2 - 4 \right)^2 \, dx
\]
\[
\frac{\sqrt{3}}{4} \left[ \frac{x^5}{5} - \frac{8x^3}{3} + 16x \right]_0^2 = \frac{256}{15}
\]

\[
\int_{\frac{\sqrt{2}}{2}}^{\sqrt{2}} \sqrt{\frac{3}{4} \left( x^2 - 4 \right)} \, dx
\]
\[
\int_{\frac{\sqrt{2}}{2}}^{\sqrt{2}} \left( x^4 - 8x^2 + 16 \right) \, dx
\]
\[
\frac{\sqrt{3}}{15} \left[ \frac{x^5}{5} - \frac{8x^3}{3} + 16x \right]_{\frac{\sqrt{2}}{2}}^{\sqrt{2}} = \frac{256}{15}
\]
\[ A = \frac{1}{2} \pi r^2 \quad r = \text{radius of cross section} \]
\[ r = \frac{1}{2} \left[ (2-x)^2 - x^2 \right] = 1-x^2 \]

* Changing radius will yield \( \frac{1}{2} \) distance between curves

Let \( R \) be the region bounded by the graph of \( y = x^2 \) and \( y = 2 - x^2 \).

Find the volume of the solid with base of region \( R \) and cross sections perpendicular to the \( x \)-axis if the cross sections are semicircles.

\[ \frac{\pi}{2} \int \left(1-x^2\right) dx \]

\[ \frac{\pi}{2} \left[ x - \frac{x^3}{3} \right] \bigg|_{-1}^{1} = \frac{8\pi}{15} \]

Homework

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- Page 431
- 60, 61
52. **Volume of a Lab Glass** A glass container can be modeled by revolving the graph of

\[ y = \begin{cases} \sqrt{0.1x^3 - 2.2x^2 + 10.9x + 22.2}, & 0 \leq x \leq 11.5 \\ 2.95, & 11.5 < x \leq 15 \end{cases} \]

about the x-axis, where x and y are measured in centimeters. Use a graphing utility to graph the function and find the volume of the container.

53. Find the volume of the solid generated if the upper half of the ellipse \(9x^2 + 25y^2 = 225\) is revolved about

(a) the x-axis to form a prolate spheroid (shaped like a football),
(b) the y-axis to form an oblate spheroid (shaped like half of a candy).

![Figure for 53(a)](image)

![Figure for 53(b)](image)

54. **Minimum Volume** The arc of \(y = 4 - (x^2/4)\) on the interval \([0, 4]\) is revolved about the line \(y = b\) (see figure).

(a) Find the volume of the resulting solid as a function of \(b\).
(b) Use a graphing utility to graph the function in part (a), and use the graph to approximate the value of \(b\) that minimizes the volume of the solid.
(c) Use calculus to find the value of \(b\) that minimizes the volume of the solid, and compare the result with the answer to part (b).

![Figure for 54](image)

55. **Water Depth in a Tank** A tank on a water tower is a sphere of radius 50 feet. Determine the depths of the water when the tank is filled to one-fourth and three-fourths of its total capacity. *(Note: Use the root-finding capabilities of a graphing utility after evaluating the definite integral.)*

56. **Modeling Data** A draftsman is asked to determine the amount of material required to produce a machine part (see figure in first column). The diameters \(d\) of the part at equally spaced points \(x\) are listed in the table. The measurements are listed in centimeters.

<table>
<thead>
<tr>
<th>(x)</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>(d)</td>
<td>4.2</td>
<td>3.8</td>
<td>4.2</td>
<td>4.7</td>
<td>5.2</td>
<td>5.7</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>(x)</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>(d)</td>
<td>5.8</td>
<td>5.4</td>
<td>4.9</td>
<td>4.4</td>
<td>4.6</td>
</tr>
</tbody>
</table>

(a) Use these data with Simpson's Rule to approximate the volume of the part.

(b) Use the regression capabilities of a graphing utility to find a fourth-degree polynomial through the points representing the radius of the solid. Plot the data and graph the model.

(c) Use a graphing utility to approximate the definite integral yielding the volume of the part. Compare the result with the answer to part (a).

57. **Think About It** Match each integral with the solid whose volume it represents, and give the dimensions of each solid.

(a) Right circular cylinder  
(b) Ellipsoid  
(c) Sphere  
(d) Right circular cone  
(e) Torus

(i) \(\int_0^r \left(\frac{rx^2}{h}\right) dx\)
(ii) \(\int_0^r r^3 dx\)
(iii) \(\int_0^r \left(\sqrt{r^2 - x^2}\right)^2 dx\)
(iv) \(\int_0^r \left(\sqrt{1 - (x/a)^2}\right)^2 dx\)
(v) \(\int_0^r \left[(R + \sqrt{r^2 - x^2})^2 - (R - \sqrt{r^2 - x^2})^2\right] dx\)

58. Find the volume of concrete in a ramp that is 3 meters wide and whose cross sections are right triangles with base 10 meters and height 2 meters (see figure).

59. Find the volume of the solid whose base is bounded by the graphs of \(y = x + 1\) and \(y = x^2 - 1\), with the indicated cross sections taken perpendicular to the x-axis.

(a) Squares  
(b) Rectangles of height 1

![Figure for 58](image)

![Figure for 59](image)
60. Find the volume of the solid whose base is bounded by the circle \( x^2 + y^2 = 4 \), with the indicated cross sections taken perpendicular to the \( x \)-axis.

(a) Squares  
(b) Equilateral triangles

(c) Semicircles  
(d) Isosceles right triangles

61. The base of a solid is bounded by \( y = x^2 \), \( y = 0 \), and \( x = 1 \). Find the volume of the solid for each of the following cross sections (taken perpendicular to the \( y \)-axis): (a) squares, (b) semicircles, (c) equilateral triangles, and (d) semicircles whose heights are twice the lengths of their bases.

62. Find the volume of the solid of intersection (the solid common to both) of the two right circular cylinders of radius \( r \) whose axes meet at right angles (see figure).

64. A manufacturer drills a hole through the center of a metal sphere of radius \( R \). The hole has a radius \( r \). Find the volume of the resulting ring.

65. For the metal sphere in Exercise 64, let \( R = 5 \). What value of \( r \) will produce a ring whose volume is exactly half the volume of the sphere?

66. The solid shown in the figure has cross sections bounded by the graph of

\[ |x|^a + |y|^a = 1 \]

where \( 1 \leq a \leq 2 \).

(a) Describe the cross section when \( a = 1 \) and \( a = 2 \).

(b) Describe a procedure for approximating the volume of the solid.

\[ |x|^1 + |y|^1 = 1 \]
\[ |x|^2 + |y|^2 = 1 \]

67. Two planes cut a right circular cylinder to form a wedge. One plane is perpendicular to the axis of the cylinder and the second makes an angle of \( \theta \) degrees with the first (see figure).

(a) Find the volume of the wedge if \( \theta = 45^\circ \).

(b) Find the volume of the wedge for an arbitrary angle \( \theta \).

Assuming that the cylinder has sufficient length, how does the volume of the wedge change as \( \theta \) increases from 0° to 90°?
33. \( \frac{\pi^2}{2} \approx 4.935 \)  
35. 1.969  
37. 49.022  
39. a

41. (a) See page 422 for the disk method.  
   (b) Horizontal axis of revolution:  
   \[ V = \pi \int_0^1 [(R(x))^2 - (r(x))^2] \, dx \]  
Vertical axis of revolution:  
\[ V = \pi \int_0^1 [(R(y))^2 - (r(y))^2] \, dy \]  
43. The parabola \( y = 4x - x^2 \) is a horizontal translation of the parabola \( y = 4 - x^2 \). Therefore, their volumes are equal.

45. 18\( \pi \)  
47. Proof  
49. \( \pi r h \left( 1 + \frac{h^2}{3H^2} \right) \)  
51. \( \frac{\pi}{30} \)

53. (a) 60\( \pi \)  
   (b) 50\( \pi \)

55. One-fourth: 32.64 feet; Three-fourths: 67.36 feet

57. (a) ii; right circular cylinder of radius \( r \) and height \( h \)  
   (b) iv; ellipsoid whose underlying ellipse has the equation \( \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \)  
   (c) iii; sphere of radius \( r \)  
   (d) i; right circular cone of radius \( r \) and height \( h \)  
   (e) v; torus of cross-sectional radius \( r \) and other radius \( R \)  

63. Proof  
65. \( 5 \sqrt{1 - 2^{-7/3}} = 3.0415 \)

67. (a) \( \frac{2\pi^2}{3} \)  
   (b) \( \frac{2\pi^3 \tan \theta}{3} \), \( \lim_{\theta \to \pi} V = \infty \)

Section 6.3 (page 437)

1. \( 2\pi \int_0^2 x^2 \, dx = \frac{16\pi}{3} \)  
2. \( 2\pi \int_0^3 x \sqrt{x} \, dx = \frac{128\pi}{5} \)
5. \( 2\pi \int_0^2 x^3 \, dx = 8\pi \)  
7. \( 2\pi \int_0^2 (4x - 2x^2) \, dx = \frac{16\pi}{3} \)
9. \( 2\pi \int_0^1 (x^3 - 4x + 4) \, dx = \frac{8\pi}{3} \)
11. \( 2\pi \int_0^1 \left( \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \right) \, dx = \sqrt{2\pi} \left( 1 - \frac{1}{\sqrt{e}} \right) = 0.986 \)
13. \( 2\pi \int_0^1 y(2 - y) \, dy = \frac{8\pi}{3} \)
15. \( 2\pi \int_0^{1/2} y \, dy + \int_{1/2}^1 y \left( \frac{1}{y} - 1 \right) \, dy = \frac{\pi}{2} \)
17. 16\( \pi \)  
19. 64\( \pi \)
21. (a) \( \frac{128\pi}{7} \)  
   (b) \( \frac{64\pi}{5} \)  
   (c) \( \frac{96\pi}{5} \)
23. (a) \( \frac{\pi a^3}{15} \)  
   (b) \( \frac{\pi a^3}{15} \)  
   (c) \( \frac{4\pi a^3}{15} \)

25. \( V = 2\pi \int_0^d p(y)h(y) \, dy \) for horizontal axis of revolution  
   \( V = 2\pi \int_a^b p(x)h(x) \, dx \) for vertical axis of revolution

27. Both integrals yield the volume of the solid generated by revolving the region bounded by the graphs of \( y = \sqrt{x - 1}, y = 0 \), and \( x = 5 \) about the \( x \)-axis.

31. (a)  
   (b) 1.506

33. \( \frac{128}{3} \)  
35. Diameter = \( 2\sqrt{4 - 2\sqrt{3}} \approx 1.464 \)

37. \( 4\pi^2 \)  
39. Proof

41. (a) ii; right circular cone of radius \( r \) and height \( h \)  
   (b) v; torus of cross-sectional radius \( r \) and other radius \( R \)  
   (c) iii; sphere of radius \( r \)  
   (d) i; right circular cone of radius \( r \) and height \( h \)  
   (e) iv; ellipsoid whose underlying ellipse has the equation \( \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \)

43. (a) 3,166,593 cubic feet  
   (b) \( d = -0.000561x^2 + 0.0189x + 19.39 \)

(c) 1,343,345 cubic feet  
(d) 10,048,221 gallons

Section 6.4 (page 447)

1. 13  
3. \( \frac{3}{2}(2\sqrt{2} - 1) \approx 1.219 \)
5. \( 5\sqrt{3} - 2\sqrt{2} \approx 8.352 \)  
7. \( \frac{31}{10} \)  
9. 1.763

11. (a)  
   (b) \( \int_0^2 \sqrt{1 + 4x^2} \, dx \approx 4.647 \)
   (c) \( \int_0^1 \sqrt{1 + \frac{1}{x^2}} \, dx \approx 2.147 \)